3.2 Shielded microstrip tranmission lines

Advanced theory

In this paragraph, we turn our attention to the computational details of the analysis of electromagnetic field distribution in a shielded microstrip transmission line (fig. **3.2B.1**) assuming constant parameters along the longitudinal axis. Then, only a two-dimensional structure has to be analyzed (cross-section of the transmission line), which simplifies computations **[20]**. For the analysis, a full-wave method, a finite-element method is exploited.

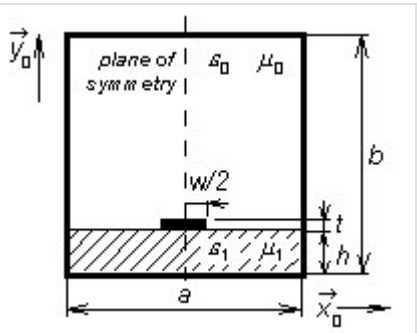

Fig. 3.2B.1 *Shielded microstrip transmission line (longitudinally homogenous)*

As already described in layer A, the analysis is based on Maxwell equations in differential form. Sources of electromagnetic waves are assumed to be in long distance from the region of analysis (then, imposed currents $\mathbf{J}_s$ are zero in this region). Charge density r in dielectrics, which surround the microstrip, is supposed to be zero. Dielectrics is expected to be isotropic and linear (permittivity and permeability are scalar quantities, which values do not depend on the value of respective intensity) and to exhibit electric losses (represented by electric conductivity s). All the metallic parts (shielding waveguide, microstrip) are assumed to be perfect electric conductors.

The analyzed microstrip transmission line is placed to Cartesian coordinate system (coordinates x and y in transversal directions, coordinate z in longitudinal one). Then, the wave can be said to propagate in the longitudinal direction z (along the microstrip), and the electric-field intensity vector depends on the longitudinal coordinate by the following way:

$$\mathbf{E}(x,\ y,\ z) = \mathbf{E}(x,\ y)\exp(-\gamma z). \qquad (3.2B.1)$$

Here, γ is the propagation constant.

Expressing all vectors as a sum of the transversal vector (index t) and the longitudinal one (index z), we get

$$\nabla_t \times(\nabla_t \times \mathbf{E}_t) - \gamma(\nabla_t E_z + \gamma \mathbf{E}_t) = k_0^2 \mu_r \tilde{\varepsilon_r} \mathbf{E}_t, \qquad (3.2B.2a)$$

$$\nabla_t \times[(\nabla_t E_z + \gamma \mathbf{E}_t) \times \mathbf{z}_0] = k_0^2 \mu_r \tilde{\varepsilon_r} E_z \mathbf{z}_0. \qquad (3.2B.2b)$$

Whereas (3.2B.2a) is vector equation for transversal components, (3.2B.2b) is scalar equation for longitudinal components. In those relations, ∇_t is a transversal operator *nabla*, $\mathbf{E}_t$ is transversal electric-field intensity vector, γ is propagation constant, E_z denotes longitudinal component of electric-field intensity, k_0 is wave number in vacuum, μ_r denotes relative permeability inside structure, $\tilde{\varepsilon_r}$ is complex relative permittivity inside structure and $\mathbf{z}_0$ denotes unitary vector in the longitudinal direction.

The set of differential equations (3.2B.2) has to be completed by boundary conditions, which have to be met by the solution of (3.2B.2)

$$\left.\begin{aligned} \mathbf{n}_0 \times \mathbf{E}_t &= 0 \\ E_z &= 0 \end{aligned}\right\} na\ \Gamma_1, \qquad (3.2B.3a)$$

$$\left.\begin{aligned} [\nabla_t E_z + \gamma \mathbf{E}_t] \cdot \mathbf{n}_0 &= 0 \\ \nabla_t \times \mathbf{E}_t &= 0 \end{aligned}\right\} na\ \Gamma_2. \qquad (3.2B.3b)$$

Eqns. (3.2B.2) completed by boundary conditions (3.2B.3) are initial relations for the full-wave analysis of the shielded microstrip transmission line. Unfortunately, the set (3.2B.2), (3.2B.3) includes the first Maxwell equation and the second one only. In order to meet the third Maxwell equation and the fourth one, the analysis has to be based on hybrid finite elements.

If the shielded microstrip transmission line is analyzed exploiting hybrid finite elements, all components of electric-field intensity vector or all components of magnetic-field one have to be included in computations. Eqn. (3.2B.2) stays the initial relation of the analysis.

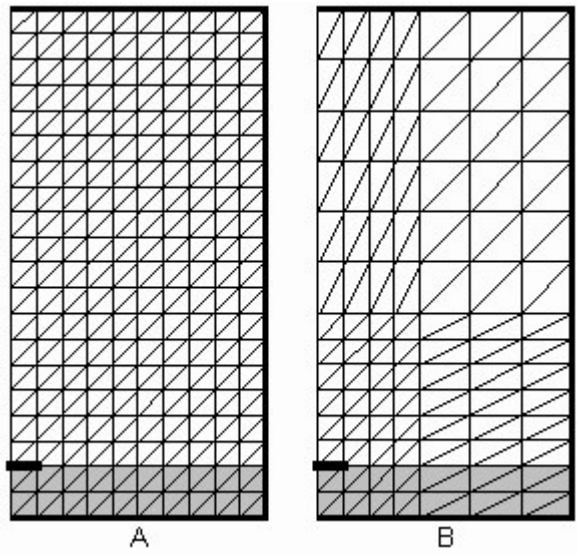

Fig. 3.2B.2 *Mesh examples of rectangular bi-elements for the analysis of a shielded microstrip transmission line*

The matter of hybrid finite elements consists in modeling the longitudinal component of field intensity exploiting nodal approximation and in modeling transversal components using edge vectors.

The first step of the finite-element method consists in dividing the analyzed structure (cross-section of the transmission line) to finite elements (non-overlapping sub-regions, which contain all the points of the analyzed structure). In the area of a finite element, parameters of the analyzed structure (permittivity, permeability, conductivity) have to be constant. There are no restrictions to shape and dimensions of finite elements. Examples of finite-element meshes are depicted in fig. **3.2B.2**.

In the second step of the solution, the distribution of a computed quantity is approximated over each finite element in a formal way. The approximation is expressed as a linear combination of elected partial approximation functions and unknown approximation coefficients.

Analyzing the shielded microstrip transmission line, a formal approximation of a scalar function $E_z = E_z(x, y)$ and a vector one $\boldsymbol{E}_t = \boldsymbol{E}_t(x, y)$ have to be expressed. Let us start with the scalar function.

The global approximation of the scalar function E_z over the whole cross-section of the transmission line is composed of local approximations over single finite elements.

Local approximation of the longitudinal component of electric-field intensity vector over a finite element is expressed as a linear combination of elected partial approximation functions and unknown approximation coefficients. Considering linear approximation, the approximation plane over a finite element is composed of three partial approximation planes. Each partial approximation plane is unitary in a unique vertex of the triangle and is zero in the other two vertexes (see fig. **3.2B.3**). Coefficients c_n at partial functions in the linear combination play the role of spatial samples of the computed function in vertexes of the finite element (fig. **3.2B.3**). Vertexes of the finite element are called nodes and respective functional values are called nodal values.

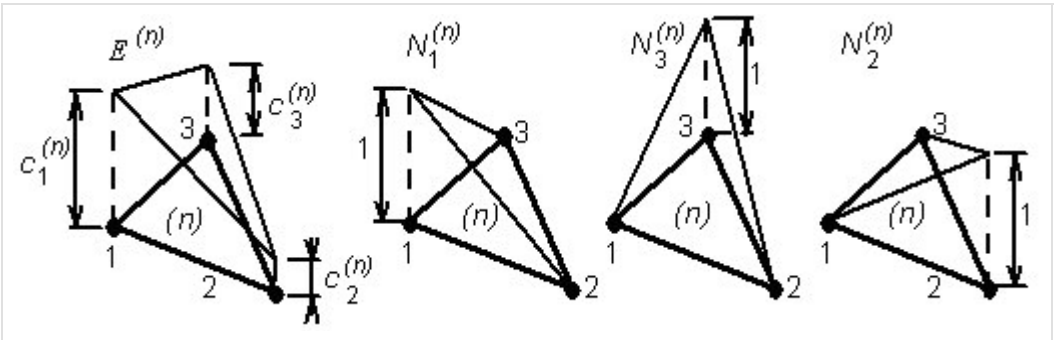

Fig. 3.2B.3 *Linear approximation of E over finite element. Composed of three shape functions*

Partial approximation functions are called shape functions. All the shape functions, which are unitary in the same node (see fig. **3.2B.4**), compose together a basis function.

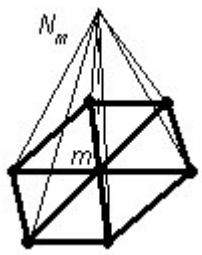

Fig. 3.2B.4 *Linear basis function related to m-th node*

In many cases, approximation functions of higher order are more suitable than the linear function. Although the triangular element has to contain more nodes (6 for quadratic approximation, 10 for cubic one, etc.), the same error comparing to linear approximation is reached even if significantly lower number of finite elements is exploited. Approximation functions of higher order are smoother, and therefore, they represent

better natural quantities.

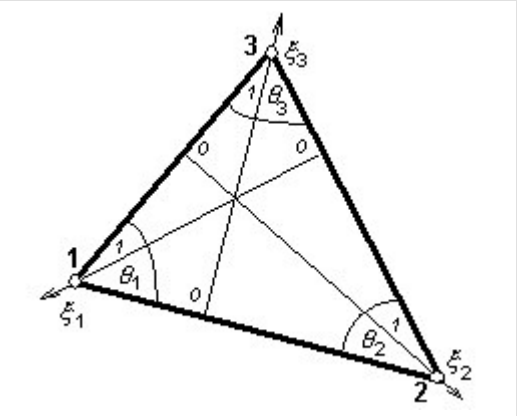

Fig. 3.2B.5 *Two-dimensional simplex coordinates*

Now, we are familiar with shape functions, and therefore, we can find their proper mathematical representation. For this purpose, Lagrange polynomials expressed in simplex coordinates **[21]** are usually used.

What does the term *simplex coordinates* mean? Considering triangular finite elements, simplex coordinate axes are of the direction of heights of the triangle. Simplex coordinates are unitary in the vertex and are zero on the opposite edge. Simplex coordinate do not depend neither on the shape nor on the dimensions of the finite element, and therefore, all the computations are sufficient to be performed once for a single finite element, and the results are recomputed for the other elements only.

Dealing with physical matter of simplex coordinates, a general point P inside a triangular finite element divides its surface to three partial triangles (fig. **3.2B.6**). The ratio of the surface of a triangle, which is positioned in front of the first node, to the surface of the whole finite element equals to the simplex coordinate of P on the first simplex axis

$$\xi_1 = \frac{\sigma(S_1)}{\sigma(S)}. \qquad (3.2B.4)$$

For other simplex coordinate axes, the situation is similar. In eqn. (3.2B.4), $\sigma(S_1)$ denotes surface of the partial triangle, which is positioned in front of the first node, and $\sigma(S)$ is surface of the whole finite element. Obviously, addition of all three simplex coordinates in an arbitrary point is unitary

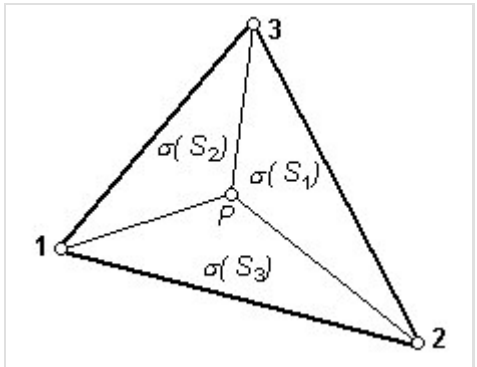

Fig. 3.2B.6 *Matter of simplex coordinates*

$$\xi_1 + \xi_2 + \xi_3 = 1. \qquad (3.2B.5)$$

As shown in **[21]**, this conclusion can be generalized for an arbitrary dimension and for an arbitrary order of an approximation polynomial.

Now, we turn our attention to Lagrange interpolation polynomials.

Lagrange polynomial of n^{th}can be expressed (using simplex coordinate x) as

$$R_m(n, \xi) = \frac{1}{m!}\prod_{k=0}^{m-1}(n\xi - k) \quad m \geq 1 \quad R_0(n, \xi) = 1. \qquad (3.2B.6)$$

Here, n is order of the approximation polynomial. Eqn. (3.2B.6) describes the whole family of polynomials: family members differ in the index m, which can vary from zero to the order of polynomial n.

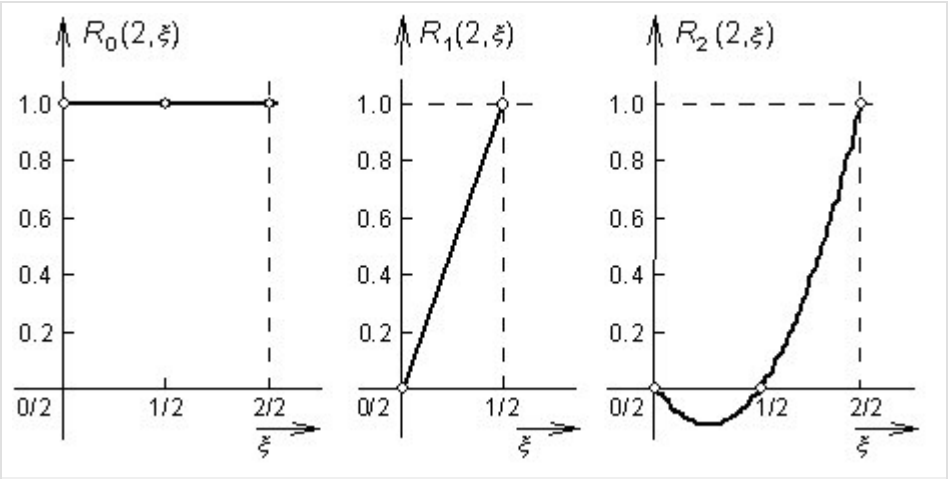

Fig. 3.2B.7 *Family of Lagrange polynomials of 2nd order*

Nulls of polynomials $R_m(n)$ are equidistantly placed on coordinates ξ= 0, 1/n till (m-1)/n, the polynomial is of unitary value in $\xi = m/n$. Hence, $R_m(n)$ is of m equidistantly placed nulls at the left from the coordinate $\xi = m/n$ and of zero nulls at the right.

The above-given statement is illustrated by fig. **3.2B.7**, where all the members of the family of quadratic polynomials R(2) are depicted. Figure demonstrates the above-described equidistant distribution of nulls. The family member of index 0, i.e. $R_0(2)$, does not have any null at the left from the coordinate 0 and is of unitary value at the coordinate 0. The family member of index 1, i.e. $R_1(2)$, is of single null at the coordinate 0 and is unitary at 1/2. Finally, the family member of index 2, i.e. $R_2(2)$, is of nulls at coordinates 0 and 1/2 and is unitary at the coordinate 1.

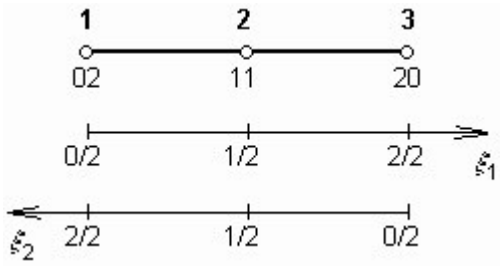

Fig. 3.2B.8 *One-dimensional finite element, its simplex coordinates and indexes of Lagrange polynomials*

Using Lagrange polynomials, we compose quadratic shape functions for one-dimensional finite element. Simplex coordinate ξ_1 is oriented from left to right on this element, the coordinate ξ_2 goes from the right to the left (fig. **3.2B.8**). The shape function related to the node 1 (unitary value in the node 1, zero value in the nodes 2 and 3) is then composed by multiplying Lagrange polynomial of variable ξ_1 and index 0 (constant function of value 1) by Lagrange polynomial of variable ξ_2 and index 2 (since the coordinate ξ_2 is oriented fro the right to the left, the course of the function $R_2(2)$ from fig. **3.2B.7** has to be reverted).

Similarly, shape functions for nodes 2 and 3 can be composed. For the node 2, Lagrange polynomials of variables ξ_1 and $\xi_2$and of index 1 are mutually multiplied. For the node 3, Lagrange polynomial of variable ξ_1 and index 2 is multiplied by Lagrange polynomial of variable ξ_2 and index 0. Indexes of Lagrange polynomials, which form shape functions of respective nodes, are written at these nodes (fig. **3.2B.8**) in the form of a fraction; numerator is an index of Lagrange polynomial of the coordinate ξ_1 and denominator is an index of Lagrange polynomial of the coordinate ξ_2. Adding numerator and denominator, order of approximation polynomial n has to be obtained.

In general, the shape function of the node (i, j) of a one-dimensional finite element can be expressed as

$$\alpha_{ij} = R_i(n, \xi_1)R_j(n, \xi_2) \quad i + j = n, \qquad (3.2B.7a)$$

here n is order of an approximation polynomial, R denotes Lagrange polynomials defined by eqn. (3.2B.5) and ξ are simplex coordinates.

In the next step, we turn our attention to a two-dimensional finite element. The only change, which has to be done, is adding a new simplex coordinate ξ_3 to two existing coordinates ξ_1 and ξ_2. Two multiplicands, which appear in relations for shape functions of a one-dimensional finite element, are completed by the third multiplicand, corresponding to Lagrange polynomial of a new simplex coordinate ξ_3

$$\alpha_{ijk} = R_i(n, \xi_1)R_j(n, \xi_2)R_k(n, \xi_3) \quad i + j + k = n. \qquad (3.2B.7b)$$

Here, ξ_1, ξ_2 and ξ_3 denote simplex coordinates of a two-dimensional finite element, n is order of an approximation polynomial and R are Lagrange polynomials.

Substituting to (3.2B.6) a (3.2B.7), we get for linear approximation the following shape functions

$$\alpha_{100} = N_1^{(n)} = \xi_1, \quad \alpha_{010} = N_2^{(n)} = \xi_2, \quad \alpha_{001} = N_3^{(n)} = \xi_3. \qquad (3.2B.8)$$

Now, we are familiar with basis functions for the approximation of a scalar function E_z. Therefore, we can turn our attention to the approximation of vector function $\mathbf{E}_t$. The approximation of a vector function formally corresponds to the approximation of a scalar function;

only basis functions are of vector nature

$$\mathbf{E}_t^{(n)} = \mathbf{N}_{t,01}^{(n)} e_{t,01}^{(n)} + \mathbf{N}_{t,12}^{(n)} e_{t,12}^{(n)} + \mathbf{N}_{t,20}^{(n)} e_{t,20}^{(n)} = \sum_{i,j} \mathbf{N}_{t,ij}^{(n)} e_{t,ij}^{(n)}. \qquad (3.2B.9)$$

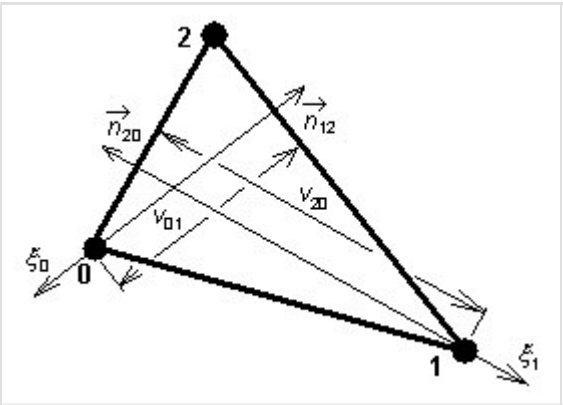

Fig. 3.2B.9 *To the explanation of behavior of vector shape function; for simplicity, superscripts (n) are missed*

In eqn. (3.2B.9) $e_{t,01}^{(n)}$ denotes the edge approximation coefficient for the approximation of the distribution of transversal components of electric-field intensity vector, which is related to the edge 0-1 of n^{th} finite element. Next, $\mathbf{N}_{t,01}^{(n)}$ denotes vector shape function, which is multiplied by the edge coefficient 0-1 in order to evaluate submission of this coefficient to the approximation of the distribution of transversal electric-field intensity vector over n^{th} finite element. Similar situation can be observed at the other two vector shape functions.

The vector shape function can be expressed as

$$\mathbf{N}_{t,ij}^{(n)} = -\xi_i \frac{l_{k,i}^{(n)}}{2A^{(n)}} \mathbf{n}_{k,i}^{(n)} + \xi_j \frac{l_{j,k}^{(n)}}{2A^{(n)}} \mathbf{n}_{j,k}^{(n)}. \qquad (3.2B.10)$$

Here, $A^{(n)}$ denotes surface of n^{th} finite element, $l_{k,i}^{(n)}$ is the length of the edge k-i of n^{th} finite element, $\mathbf{n}_{k,i}^{(n)}$ is the normal to the edge k-i of n^{th} finite element and ξ_i is a classic simplex coordinate, which is unitary in node i and which is zero in the opposite edge. The meaning of the rest of symbols is similar.

Let us observe behavior of the shape function (2.3B.10) in node 0. Here, the simplex coordinate ξ_0 is of unitary value and ξ_1 is zero. In node 0, the shape function (2.3B.10) is perpendicular to the edge 2-0 and is oriented inside the finite element (due to the negative sign). Its magnitude equals to the reverse value of the height v_{20}.In node 1, shape function is of the direction of the normal to the edge 1-2 and its magnitude equals to the reverse value of the height v_{01}. Moving from node 0 to node 1 along the edge 0-1, the direction of the shape function (2.3B.10) continuously changes from $-\mathbf{n}_{01}^{(n)}$ to $+\mathbf{n}_{12}^{(n)}$ and its magnitude changes from the value $(1/v_{20})$ to $(1/v_{01})$. Since the shape function (2.3B.10) does not depend on the coordinate ξ_2, the described behavior is the same along all the parallels of the edge 0-1.

Now, all the components of the electric-field intensity vector are approximated in a formal way. Therefore, the next step of the finite-element method can be done: the formal approximation is substituted to the solved equation, and an approximation error (the difference between the approximate solution and the exact one) is minimized. Performing these steps, we obtain the final matrix equation:

$$\begin{bmatrix} \mathbf{S}_t^{(n)} - k_0^2 \mu_r^{(n)} \tilde{\varepsilon}_r^{(n)} \mathbf{T}_t^{(n)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_t^{(n)} \\ \mathbf{E}_z^{(n)} \end{bmatrix} = \gamma^2 \begin{bmatrix} \mathbf{T}_t^{(n)} & \mathbf{G}^{(n)} \\ \mathbf{G}^{(n)\mathrm{T}} & \mathbf{S}_z^{(n)} - k_0^2 \mu_r^{(n)} \tilde{\varepsilon}_r^{(n)} \mathbf{T}_z^{(n)} \end{bmatrix} \begin{bmatrix} E_t^{(n)} \\ E_z^{(n)} \end{bmatrix},$$

where

$$\mathbf{T}_t^{(n)} \mathbf{E}_t^{(n)} = \sum_{i,j} \left\{ e_{t,ij}^{(n)} \iint_{S^{(n)}} \left[\mathbf{N}_{t,rs}^{(n)} \cdot \mathbf{N}_{t,ij}^{(n)} \right] dS \right\}, \qquad (3.2B.11a)$$

$$\mathbf{G}^{(n)} \mathbf{E}_z^{(n)} = \sum_{m} \left\{ e_{z,m}^{(n)} \iint_{S^{(n)}} \left[\mathbf{N}_{t,rs}^{(n)} \cdot \left(\nabla_t N_{z,m}^{(n)} \right) \right] dS \right\}, \qquad (3.2B.11b)$$

$$\mathbf{S}_z^{(n)} \mathbf{E}_z^{(n)} = \sum_{m} \left\{ e_{z,m}^{(n)} \iint_{S^{(n)}} \left[\left(\nabla_t N_{z,q}^{(n)} \right) \cdot \left(\nabla_t N_{z,m}^{(n)} \right) \right] dS \right\}, \qquad (3.2B.11c)$$

$$\mathbf{T}_z^{(n)}\,\mathbf{E}_z^{(n)} = \sum_m \left\{ e_{z,m}^{(n)} \iint_{S^{(n)}} \left[N_{z,q}^{(n)} N_{z,m}^{(n)} \right] dS \right\}, \qquad (3.2B.11d)$$

$$\mathbf{S}_t^{(n)}\,\mathbf{E}_t^{(n)} = \sum_{i,j} \left\{ e_{t,ij}^{(n)} \iint_{S^{(n)}} \left[\left(\nabla_t \times \mathbf{N}_{t,rs}^{(n)} \right) \cdot \left(\nabla_t \times \mathbf{N}_{t,ij}^{(n)} \right) \right] dS \right\}. \qquad (3.2B.11e)$$

In the above-given relations, $\mathbf{E}_t^{(n)}$ is column vector of three unknown edge approximation coefficient (related to the approximation of transversal components of electric-field intensity vector) over n^{th} finite element and $\mathbf{E}_z^{(n)}$ denotes column vector of three unknown nodal approximation coefficients (related to the approximation of longitudinal component of electric-field intensity vector) over n^{th} finite element. Next, γ denotes complex propagation constant propagation constant, k_0 is wave number in vacuum, $\mu_r^{(n)}$ is relative permeability of n^{th} finite element and $\tilde{\varepsilon}_r^{(n)}$ is complex relative permittivity of the same element. Symbol dS denotes elementary facet for the integration over n^{th} finite element and $S^{(n)}$ is the total surface of n^{th} finite element. Summation over the index mrepresents addition over all nodes of the finite element (i.e., m = 0, 1, 2) and summation over indexes i, j represents addition over all edges of the element (i.e., i, j = 0-1, 1-2, 2-0). Symbols $e_{t,\,ij}^{(n)}$ represent edge approximation coefficients, symbols $e_{z,\,m}^{(n)}$ represent nodal approximation coefficients.

Matrices $\mathbf{T}_t^{(n)}$, $\mathbf{G}^{(n)}$, $\mathbf{S}_z^{(n)}$, $\mathbf{T}_z^{(n)}$ and $\mathbf{T}_t^{(n)}$ are matrices of coefficients of n^{th} finite element of the size 3 x 3. Elements of the above-described matrices were computed by the integration of the product of basic functions and weighting ones (or their derivatives) over n^{th} finite element (in simplex coordinates, of course). Those matrices can be evaluated exploiting the following relations:

$$\mathbf{S}_t^{(n)} = \frac{1}{A^{(n)}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad (3.2B.12a)$$

$$\mathbf{T}_t^{(n)} = \frac{1}{12} \sum_{i=0}^{2} \mathbf{Q}_i \, cotg\left[\theta_i^{(n)} \right], \qquad (3.2B.12b)$$

$$\mathbf{G}^{(n)} = \frac{1}{6} \sum_{i=0}^{2} \mathbf{C}_i \, cotg\left[\theta_i^{(n)} \right], \qquad (3.2B.12c)$$

$$\mathbf{S}_z^{(n)} = \frac{1}{2} \sum_{i=0}^{2} \mathbf{D}_i \, cotg\left[\theta_i^{(n)} \right], \qquad (3.2B.12d)$$

$$\mathbf{T}_z^{(n)} = \frac{A^{(n)}}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad (3.2B.12e)$$

where

$$\mathbf{Q}_0 = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad \mathbf{Q}_1 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad \mathbf{Q}_2 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix},$$

$$\mathbf{C}_0 = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{C}_1 = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ -2 & 2 & 0 \end{bmatrix},$$

$$\mathbf{D}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$A^{(n)}$ is surface of n^{th} finite element and $\theta_i^{(n)}$ is angle at i^{th} vertex of n^{th} finite element. The above-given relations are valid for the following organization of nodes and edges:

$$\mathbf{E}^{(n)} = \begin{bmatrix} e_{z,0}^{(n)} & e_{z,1}^{(n)} & e_{z,2}^{(n)} & e_{t,12}^{(n)} & e_{t,20}^{(n)} & e_{t,01}^{(n)} \end{bmatrix}^T. \qquad (3.2B.12f)$$

Solving the matrix equation for the vector of unknown approximation coefficients, the solution of the problem is obtained. Substituting approximation coefficients to the formal approximation, a real approximation of a sought function in every point of n^{th} finite element is obtained. Associating approximations over all finite elements, a global approximation of the solution of the partial differential equation in all the points of the analyzed structure is found.

In layer C, a matlab program analyzing a shielded microstrip transmission line by the described finite-element method is introduced. A practical programmer's description is given in the layer D.